Theory of Computation
Reducibility

Dimitris Diochnos
School of Computer Science
University of Oklahoma

Outline

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Introduction

Oftentimes we reduce the solution of one problem to the solution of another problem.

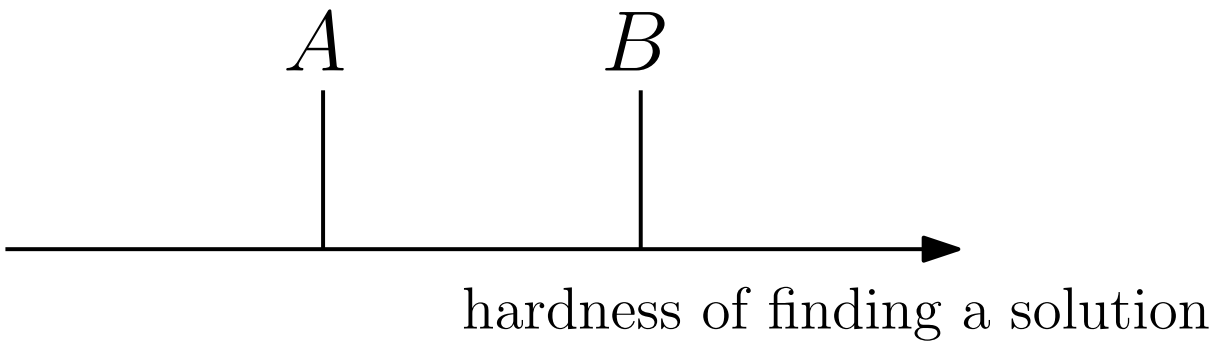
For example,

measure the area of a rectangle $\xrightarrow{\text{reduces to}}$

measure the length and height of a rectangle

Introduction

"Problem A is reducible to Problem B" implies:



- A solution to B provides a solution to A in this case. So A can not be harder than B.
- Conversely: If problem A is "hard" and A is reducible to B, then B is also "hard" (or "harder"...)

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Undecidable Problems from Language Theory

Theorem 1

$$HALT_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on input string } w\}$$

is undecidable.

Undecidable Problems from Language Theory

Theorem 1

$$HALT_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on input string } w\}$$

is undecidable.

Proof.

Towards contradiction, assume HALT_{TM} is decidable.
Then, there is a TM R that decides HALT_{TM}.
We now reduce A_{TM} to HALT_{TM}:
Let S be a TM such that: "on input $\langle M, w\rangle$, where M is a TM and w is a string, does the following:

1. Run R on input $\langle M, w\rangle$
   If "no" (i.e., $\langle M, w\rangle$ rejected) $\Rightarrow$ M loops forever on $w \Rightarrow$ **reject**.
2. If "yes" from (1), then run M on w and return whatever M returns."
   (Impossible to have such an S because now we decide A_{TM}) □

Undecidable Problems from Language Theory

Theorem 2

$$E_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } M \text{ and } L(M) = \emptyset\}$$

is undecidable.

Undecidable Problems from Language Theory

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is undecidable.

Proof (to be cont.)

Towards contradiction, assume E_{TM} is decidable by some TM R.
We want to reduce A_{TM} to E_{TM}.
(Technically we will reduce A_{TM} to $\overline{\mathrm{E}_{TM}}$ since M accepts if $L(M) \neq \emptyset$.)
For the critical input $\langle M, w \rangle$ of A_{TM}, construct another TM M_1:
$M_1 =$ "on input x:

1. If $x \neq w$, **reject**.
2. If $x = w$, run M on input w and accept if M accepts." (cont'd $\rightarrow$)

Undecidable Problems from Language Theory

Proof (cont.)

So, $L(M_1)$ is either $\emptyset$ or $\{w\}$.
Check with R which case is true.
Construct another TM S such that:
$S =$ "on input $\langle M, w \rangle$, an encoding of a TM M and a string w:

1. Use the description of M and w to construct the TM M_1.
2. Run R on input $\langle M_1 \rangle$.
3. If R accepts, **reject**,
   if R rejects, **accept**."

In other words, S is now a decider for A_{TM}.
This is impossible, so we reach a contradiction. □

Undecidable Problems from Language Theory

Theorem 3

$$EQ_{TM} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$$

is undecidable.

Undecidable Problems from Language Theory

Theorem 3

$$EQ_{TM} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$$

is undecidable.

Proof.

Towards contradiction, assume EQ_{TM} is decidable by some TM R.
Reduce E_{TM} to EQ_{TM}. The idea is

- to create a TM M_1 that rejects all inputs, and
- use R to decide $\langle M, M_1\rangle$ for some arbitrary M, thus solving E_{TM} (which is impossible).

$S =$ "on input $\langle M\rangle$, where M is a TM:

1. Run R on input $\langle M, M_1\rangle$, where M_1 is a TM that rejects all inputs.
2. If R accepts, **accept**. If R rejects, **reject**."

$\Rightarrow$ So S decides $\mathrm{E}_{TM} \Rightarrow$ contradiction. □

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The Post Correspondence Problem

We have dominoes that look like:

$$\left[\frac{a}{ab}\right]$$

A collection of dominoes looks like:

$$\left\{\left[\frac{b}{ca}\right], \left[\frac{a}{ab}\right], \left[\frac{ca}{a}\right], \left[\frac{abc}{c}\right]\right\}$$

The task is to make a list of these dominoes (repetitions permitted) so that we get the same string at the top and the bottom.
Such a list is called a **match**. For example,

$$\left[\frac{a}{ab}\right], \left[\frac{b}{ca}\right], \left[\frac{ca}{a}\right], \left[\frac{a}{ab}\right], \left[\frac{abc}{c}\right]$$

The Post Correspondence Problem

Theorem 4

$PCP = \{\langle P \rangle \mid P \text{ is an instance of the Post correspondence problem with a match}\}$

is undecidable.

Proof idea.

Reduction from A_{TM} to PCP via accepting computation histories. □

Rice's Theorem

Problem 5 (Rice's Theorem)

Let P be a property about the language recognized by a Turing machine such that:

(i) *P is non-trivial. Some, but not all Turing machines recognize a language that has this property,*

(ii) *whenever $L(M_1) = L(M_2)$ for two Turing machines M_1 and M_2, then both $L(M_1)$ and $L(M_2)$ have the property P (or none of them has the property P),*

then the language
$L_P = \{\langle M \rangle \mid M \text{ is a TM such that } L(M) \text{ recognized by } M \text{ has property } P\}$ *is undecidable.*

Rice's Theorem

Proof idea (to be cont.)

Towards contradiction, suppose L_P is decidable by some TM R.

- By condition (i), since P is a non-trivial property for some language, there are TM $M_P, M_{\neg P}$ such that:
  $$\begin{cases} L(M_P) & \text{has the property } P \\ L(M_{\neg P}) & \text{does not have the property } P \end{cases}$$
  We assume that we have access to such TMs.
- We will now reduce A_{TM} to L_P.
  Eventually our argument depends on whether or not the empty language $L_\emptyset = \emptyset$ has this property P or not.
  We construct a TM S such that:
  $$L(S) = \begin{cases} L_\emptyset, & \text{if } M \text{ does not accept } w \\ \begin{cases} L(M_P), & \text{if } M \text{ accepts } w \text{ and } L_\emptyset \text{ doesn't have property } P \\ L(M_{\neg P}), & \text{if } M \text{ accepts } w \text{ and } L_\emptyset \text{ does have the property } P \end{cases} \end{cases}$$

Rice's Theorem

Proof idea (cont.)

- We feed the description of S, $\langle S \rangle$, into R
  $\Rightarrow$ R tells us if $L(S)$ has the property
  $\Rightarrow$ Based on the relationship of $L(S)$ with $L_\emptyset$ we can tell if M accepts w
  $\Rightarrow$ **Contradiction**
- Case where $L_\emptyset = \emptyset$ does not have property P
  $S =$ "on input x:
  1. For the moment ignore x and simulate M on input w, until M accepts so that we can go to step 2.
     (So, if M rejects, loop and try again, ...forever ...)
  2. (Reached this point because M accepted w)
     Simulate M_P on x, if $L_\emptyset$ does not have the property P.
     Otherwise,
     Simulate $M_{\neg P}$ on x ($L_\emptyset$ has the property P this time).
     Accept x, if the simulation by M_P (or $M_{\neg P}$) accepted."

Rice's Theorem

Proof idea (cont.)

In other words:

M accepts $w \Rightarrow L(S) = L(M_P)$ (or $L(S) = L(M_{\neg P})$)

M does not accept $w \Rightarrow L(S) = L_\emptyset$

- Feed $\langle S \rangle$ into R

$$R(\langle S \rangle) = \begin{cases} \text{accept} & \text{if } L_\emptyset \text{ has property } P, \text{ then } L(S) = L_\emptyset \\ & \Rightarrow M \text{ did not accept } w \\ & \Rightarrow \text{For the Problem } A_{TM} \text{ answer } \textbf{reject} \\ & \quad \text{on input } \langle M, w \rangle \\ \text{reject} & \text{if } L_\emptyset \text{ does not have property } P, \text{ then } L(S) = L(M_P) \\ & \Rightarrow M \text{ accepted } w \\ & \Rightarrow \text{For the Problem } A_{TM} \text{ answer } \textbf{accept} \\ & \quad \text{on input } \langle M, w \rangle \end{cases}$$

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Mapping Reducibility

Definition 6

A function $f : \Sigma^* \to \Sigma^*$ is a **computable function** if some Turing machine M, on every input w, halts with just $f(w)$ on its tape.

Example 7

All usual arithmetic operations on integers are computable functions.
For example, addition: Input $\langle m, n \rangle$, output $\langle m + n \rangle$.

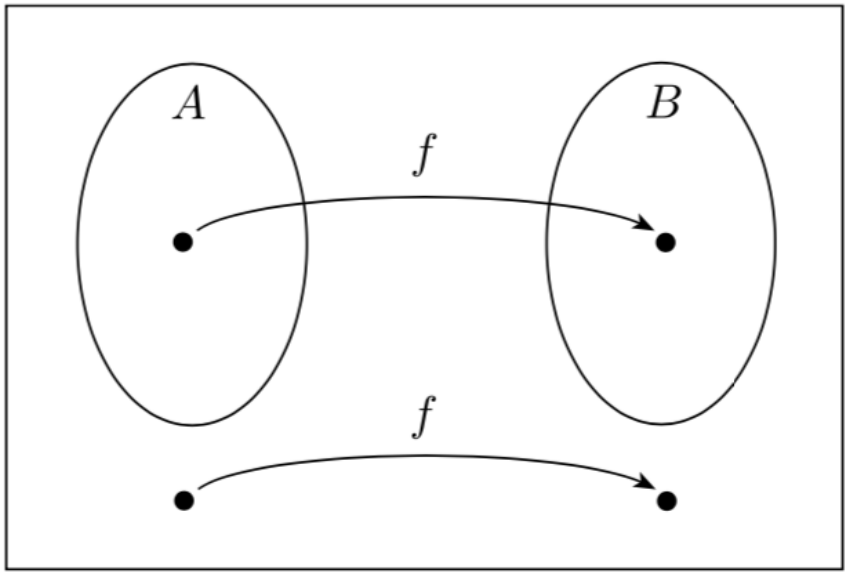
Mapping Reducibility

Definition 8

A language A is mapping reducible to language B, written $A \leq_m B$, if there is a computable function $f : \Sigma^* \to \Sigma^*$, where for every w,

$$w \in A \Leftrightarrow f(w) \in B$$

The function f is called the **reduction** of A to B.



Function f reducing A to B

As usual, computational problems are represented as languages.

Mapping Reducibility

Theorem 9

If $A \leq_m B$ and B is decidable, then A is decidable.

Proof.

Let M be a decider for B. Let f be the reduction from A to B.
Construct a decider N for A as follows.
$N =$ "On input w:

1. Compute $f(w)$.
2. Run M on input $f(w)$ and output whatever M outputs."

$$w \in A \Rightarrow f(w) \in B \quad \Rightarrow M \text{ accepts } f(w) \text{ whenever } w \in A$$
$$\Rightarrow N \text{ works as expected.}$$

Mapping Reducibility

Corollary 10

If $A \leq_m B$ and A is undecidable, then B is undecidable.

Mapping Reducibility

Example 11

In Theorem 1 we reduced A_{TM} to $HALT_{TM}$.
We need a function

$$f : \langle M, w \rangle \in A_{TM} \longrightarrow \langle M', w' \rangle \in \text{HALT}_{TM}$$

where $\langle M, w \rangle \in A_{TM}$ if and only if $\langle M', w' \rangle \in \text{HALT}_{TM}$.
$F =$ "On input $\langle M, w \rangle$:

1. Construct the TM M':
   $M' =$ "On input x:
   1. Run M on x.
   2. If M accepts, accept.
   3. If M rejects, enter a loop."
2. Output $\langle M', w \rangle$."

Mapping Reducibility

Remark 1 (malformed inputs)

When we describe a TM that computes a reduction from A to B, improperly formed inputs are assumed to map to strings outside of B.

Example 12

In Theorem 4 we reduced E_{TM} to EQ_{TM}. There the reduction f maps $\langle M \rangle$ to $\langle M, M_1 \rangle$, where M_1 is the machine that rejects all inputs.

Example 13

In Theorem 2 we reduced A_{TM} to $\overline{E_{TM}}$, since M accepts w **iff** $L(M_1)$ is **not** empty. So f is a mapping reduction from A_{TM} to $\overline{E_{TM}}$.
In fact, no reduction from A_{TM} to E_{TM} exists.

Mapping Reducibility

Theorem 14

If $A \leq_m B$ and B is Turing-recognizable, then A is Turing-recognizable.

Proof.

Same as in Theorem 9, except that M and N are recognizers instead of deciders. □

Mapping Reducibility

Corollary 15

If $A \leq_m B$ and A is not Turing-recognizable, then B is not Turing-recognizable.

Typical application of Corollary 15:

- A mapping reduction $f : A \leq_m B$ is also a mapping reduction $f : \overline{A} \leq_m \overline{B}$.
- So, since we know that $\overline{A_{TM}}$ is not Turing-recognizable, we can show that **a problem** P is **not Turing-recognizable** by designing a mapping reduction $f : A_{TM} \leq_m \overline{P}$. (This will also imply $\overline{A_{TM}} \leq_m P$.)

Mapping Reducibility

Theorem 16

$EQ_{TM} = \{\langle M_1, M_2\rangle \mid M_1$ and M_2 are TMs and $L(M_1) = L(M_2)\}$ is neither Turing-recognizable nor co-Turing-recognizable.

Proof (to be cont.)

Part A: "EQ_{TM} is not Turing-recognizable"

A_{TM} reduces to $\overline{EQ_{TM}}$.
$F =$ "On input $\langle M, w\rangle$, where M is a TM and w is a string:

1. Construct the following two machines, M_1 and M_2:
   $M_1 =$ "On any input, **reject**."
   $M_2 =$ "On any input:
   Run M on w. If M accepts, **accept**."
2. Output $\langle M_1, M_2\rangle$."

Mapping Reducibility

Proof (to be cont.)

Here, M_1 accepts nothing.
$\{\Sigma^*\} = L(M_2) \neq L(M_1) = \emptyset$, if M accepts w.
$L(M_2) = L(M_1) = \emptyset$, if M does not accept w.

So, M_1 and M_2 are **equivalent** in the second case
$\Rightarrow F$ reduces A_{TM} to $\overline{EQ_{TM}}$, as desired.

Mapping Reducibility

Proof (to be cont.)

Part B: "$\overline{EQ_{TM}}$ is not Turing-recognizable"

A_{TM} reduces to EQ_{TM}.
$G =$ "On input $\langle M, w \rangle$, where M is a TM and w is a string:

1. Construct the following two machines, M_1 and M_2:
   $M_1 =$ "On any input, **accept**."
   $M_2 =$ "On any input:
   Run M on w. If M accepts, **accept**."
2. Output $\langle M_1, M_2 \rangle$."

Mapping Reducibility

Proof (cont.)

Here, M_1 accepts everything.
This time:
M accepts $w \Rightarrow L(M_2) = L(M_1)$
M does not accept $w \Rightarrow L(M_2) \neq L(M_1)$

So, M_1 and M_2 are **equivalent** in the first case (where M accepts w)
$\Rightarrow$ G reduces A_{TM} to EQ_{TM}, as desired. □

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Turing Reducibility

Definition 17

An **oracle** for a language B is an external device that is capable of reporting whether any string w is a member of B.

An **oracle Turing machine** is a modified Turing machine that has the additional capability of querying an oracle. We write M^B to describe an oracle Turing machine that has an oracle for language B.

Turing Reducibility

Example 18

An oracle Turing machine with an oracle for A_{TM} can decide more languages than an ordinary Turing machine can.
Apart from deciding A_{TM} (obviously), we can also decide, eg., E_{TM}, the emptiness testing problem for TMs by using $T^{A_{TM}}$ below.

$T^{A_{TM}}$ = "On input $\langle M \rangle$, where M is a TM:

1. Construct the following TM N:
   N = "On any input x:
   1. Run M in parallel on all strings in Σ^*.
   2. If M accepts any of these strings, accept x."
2. Query the oracle for A_{TM} to determine whether $\langle N, 0 \rangle \in A_{TM}$.
3. If the oracle answers NO, **accept**; otherwise, **reject**."

Turing Reducibility

Example 18 (cont.)

- So, if $L(M) \neq \emptyset \Rightarrow L(N) = \{w \in \Sigma^*\} \Rightarrow 0 \in L(N) \Rightarrow$ Oracle "*Yes*" $\Rightarrow$ reject.
- Conversely, if $L(M) = \emptyset \Rightarrow L(N) = \emptyset \Rightarrow 0 \notin L(N) \Rightarrow$ Oracle "*No*" $\Rightarrow$ accept.

In other words, $T^{A_{TM}}$ decides E_{TM}.

We say that E_{TM} is **decidable relative to** A_{TM}.

Turing Reducibility

Definition 19

Language A is **Turing reducible** to language B, written $A \leq_T B$, if A is decidable relative to B.

Theorem 20

If $A \leq_T B$ and B is decidable, then A is decidable.

Proof.

B is decidable $\Rightarrow \exists$ TM M that decides $B \Rightarrow$ Use M instead of an oracle for B in order to solve B and subsequently A. $\square$